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DETERMINATION OF THE UNDERGROUND CONTOUR OF A SUBMERGED APRON WITH A REGION OF CONSTANT VELOCITY WHEN THERE IS A SALT BACKWATER†

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Within the framework of two-dimensional seepage theory, the underground contour of a submerged apron with a region of constant velocity in the case where there is a layer of stagnant salt water under the apron is constructed. The solution of the corresponding boundary-value problem is found by Polubarinova–Kochina's method [1] using the results obtained in [2]. The results of numerical calculations are given and the influence of the fundamental defining parameters of the model on the shape and size of the underground contour of the apron is analysed. Mention is made of special and limiting cases: a scheme with a water-confining stratum [3], an unsubmerged apron [2] and flow around a tongue [4, 5]. © 1998 Elsevier Science Ltd. All rights reserved.

1. STATEMENT OF THE PROBLEM

Consider the steady plane flow of fresh water of density ρ_1 under an underground impermeable contour of a submerged apron in the case where a layer of salt water of density $\rho_2 (\rho_2 > \rho_1)$ appears at a certain depth above an impermeable layer of salt. It is assumed that the soil is homogeneous and the motion obeys Darcy's law with know seepage coefficient $\varkappa = \text{const}$. Let the contour of the foundation of the apron BC consist of two vertical segments BB_1 and CC_1 and a curvilinear part B_1C_1 with constant flow velocity $|\upsilon| = \upsilon_0$ (Fig. 1). The pressure H acting on the installation, the lengths of the vertical segments d_1 and d_2, and also the width $l = l_1 + l_2$ of the apron, the left-hand end of which is fixed at the point B, are assumed given and the boundaries of the head and tail AB and CD are horizontal.

The seepage zone z (Fig. 1) is bounded below by the line AD, which passes through the fixed point $z = -ih_0$, where h_0 is the initial depth of the salt water layer (before squeezing). Allowing for the immobility of the salt water as well as the continuity of the pressure on crossing the boundary AD and using the fact that $h_0 = (h_1 + h_2)/2$, we obtain [1, p. 332]

$$h_1 = h_0 + \frac{H}{2\rho}, \quad h_2 = h_0 - \frac{H}{2\rho}, \quad \rho = \frac{\rho_2}{\rho_1} - 1 \tag{1.1}$$

where h_1 and h_2 are the depths of water flow at infinity to the left and right, respectively.

Thus, the region of ground water flow becomes fully defined but is not known in advance.

We now introduce the complex potential of motion $\omega = \varphi + i\psi$ (the domain of variation of the variable ω is shown in Fig. 2) and the complex coordinate $z = x + iy$, respectively relative to $\varkappa h_0$ and h_0. The problem is to determine the position of the curve B_1C_1 and the boundary of separation AD under the boundary conditions

$$\begin{aligned}
&AB{:}y = 0,\ \varphi = -H/2;\quad BB_1{:}x = -l_1,\ \psi = Q \\
&B_1C_1{:}\psi = Q;\quad C_1C{:}x = l_2,\ \psi = Q \\
&CD{:}y = 0,\quad \varphi = H/2;\quad AD{:}\psi = 0,\ \varphi - \rho y = \text{const}
\end{aligned} \tag{1.2}$$

so that the seepage rate along the curvilinear part of the underground contour of the apron B_1C_1 has a given constant value v_0. In addition, the seepage flow rate Q must be found.

2. CONSTRUCTION OF THE SOLUTION

The problem is solved by Polubarinova–Kochina's method based on the use of the analytic theory of linear differential equations [1]. We introduce an auxiliary variable ξ, the range of variation of which is shown in Fig. 3(a), and the required functions

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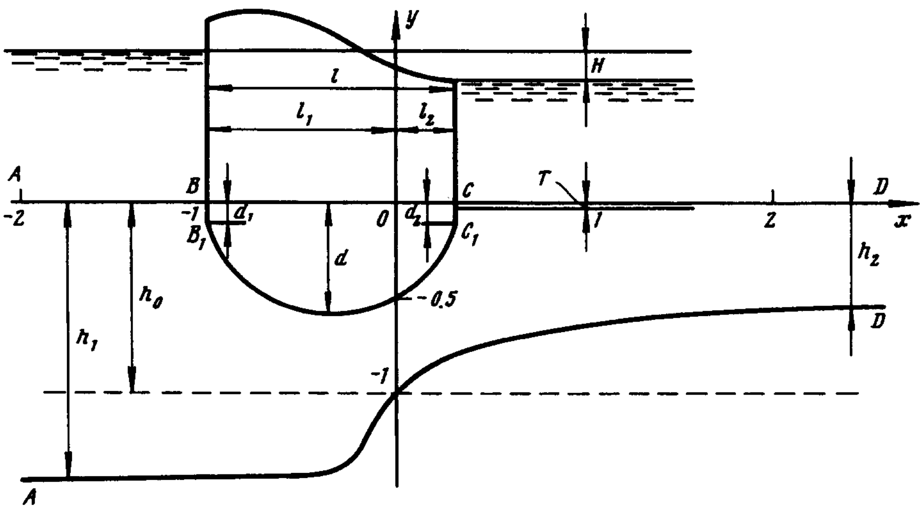

Fig. 1.

$$Z = dz/d\xi, \quad F = d\omega/d\xi \tag{2.1}$$

The region of the complex velocity $w = d\omega/dz = F/Z$ corresponding to boundary conditions (1.2) is shown in Fig. 4 and is a circular pentagon with right angles at the vertices A, D, C_1, B_1 and the cut AGD. It belongs to the class of polygons in polar grids [6, p. 175], that is, it is bounded by arcs of concentric circles and segments of straight lines passing through the origin of coordinates.

Unlike the usual method in which these polygons are transformed into rectilinear ones by means of a logarithmic function and the Christoffel–Schwarz formula is then applied, we propose to use the theory of the construction of mapping functions based on the solution of the Fuchs equations [1, 7–12]. Then the conformal mapping can be carried out by elementary methods at once in closed forms (in terms of special, and in a number of cases elementary, functions), making it simple and convenient for practical purposes; all the necessary constants of the mapping are determined incidentally when constructing the solution.

In this case, in order to map conformally the upper half-plane of ξ onto the circular pentagon of the w plane (Fig. 4), we need to construct two linearly independent integrals of the following linear second-order differential equation of the Fuchs class with five singular points [1, 6, 10]

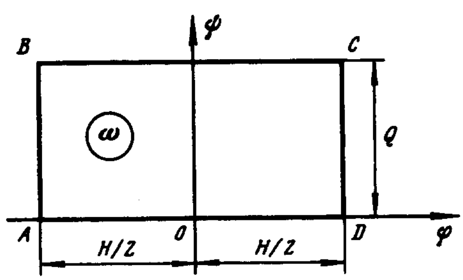

Fig. 2.

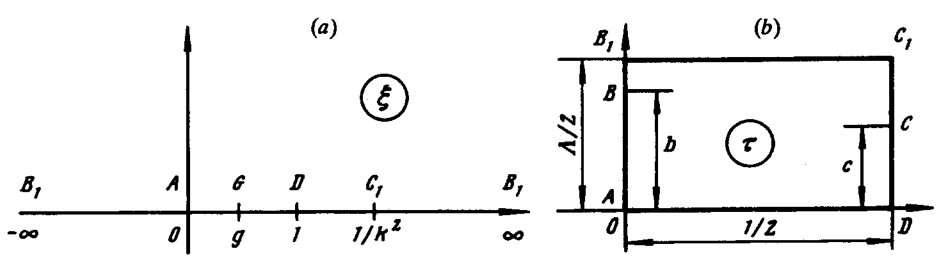

Fig. 3.

$$y''+\left[\frac{1}{2}\left(\frac{1}{\xi}+\frac{1}{\xi-1}+\frac{1}{\xi-k^{-2}}\right)-\frac{1}{\xi-g}\right]y'+\frac{\lambda_1\xi+\lambda_0}{4\xi(\xi-1)(\xi-k^{-2})(\xi-g)}y=0 \tag{2.2}$$

The constant k, inverse image $\xi = g$ of the vertex of the cut G, and also the additional parameters λ_0 and λ_1 remain undefined in the statement of the problem.

The change of variables

$$\xi = \mathrm{sn}^2(2K\tau, k) \tag{2.3}$$

where sn is the Jacobi elliptic sine with modulus k [14], transforms the upper half-plane of ξ into the rectangle $0 < \mathrm{Re}\,\tau < 1/2$, $0 < \mathrm{Im}\,\tau < \Lambda/2$ in the τ plane (Fig. 3b), where $\Lambda = K'/K$, $K' = K(k')$ and $k' = \sqrt{(1-k^2)}$, $K(k)$ is a complete first-order elliptical integral of the first kind.

Using the techniques for integrating equations of this type, devised in [12, 13], we obtain

$$y_1 = \vartheta_4(\tau-\alpha)/\vartheta_4(\tau), \quad y_2 = \vartheta_4(\tau+\alpha)/\vartheta_4(\tau) \tag{2.4}$$

where α is some suitable constant and ϑ_4 is the theta-function.

The function which maps conformally a rectangle in the τ plane into a given circular pentagon in the w plane must be expressed in terms of the ratio of linear combinations of the solutions y_1 and y_2. If we compare these combinations and use the correspondence of the points A, D, B_1 and C_1 in the τ and w planes, we find for $\upsilon_0 < \rho$

$$w(\tau)=\upsilon_0\frac{\vartheta_4(\tau+\alpha)-\vartheta_4(\tau-\alpha)}{e^{-i\pi\beta}\vartheta_4(\tau+\alpha)+e^{i\pi\beta}\vartheta_4(\tau-\alpha)} \tag{2.5}$$

$$\alpha=\pi^{-1}\operatorname{arctg}\sqrt{(\rho+\upsilon_0)/(\rho-\upsilon_0)},\quad \beta=\pi^{-1}\operatorname{arctg}\left(\upsilon_0/\sqrt{\rho^2-\upsilon_0^2}\right)$$

Defining the indices of functions (2.1) near singular points [1, 10] and allowing for relations (2.3) and (2.5), we arrive at the dependences

$$\frac{d\omega}{d\tau}=f(\tau),\quad \frac{dz}{d\tau}=\frac{f(\tau)}{w(\tau)},\quad f(\tau)=\frac{2HK(k)\mathrm{dn}(2K\tau,\ k)}{K(\lambda)\sqrt{\Delta_B\Delta_C}} \tag{2.6}$$

$$\Delta_B=(1-B^2)\mathrm{sn}^2(2K\tau,\ k)+B^2,\quad \Delta_C=1-(1-k'^2C^2)\mathrm{sn}^2(2K\tau,\ k)$$

$$B=\mathrm{sn}(2Kb,\ k'),\quad C=\mathrm{sn}(2Kc,\ k')$$

$$\lambda=\sqrt{1-(k'BC)^2}$$

The seepage flow rate is given by the formula

$$Q = \pi K(\lambda')/K(\lambda) \tag{2.7}$$

It can be verified that the functions (2.1) defined by relations (2.6) and (2.3) satisfy boundary conditions (1, 2) written in terms of the functions mentioned and, therefore, are a parametric solution of the original boundary-value problem.

3. SPECIAL AND LIMITING CASES; THE CRITICAL REGIME

1. *The case* $\rho = \infty$ *(a scheme with a backwater)*. We will first discuss the limiting case $\rho = \infty$ ($\rho_2 = \infty$) which, within the framework of the given seepage model, can be interpreted as the "freezing" of salt water. The boundary of separation is converted into a horizontal backwater, as we can see, using Eq. (2.5) and the expression for β and allowing for the fact that when $\rho = \infty$ on AD

$$\frac{dz}{d\tau}=\frac{f(\tau)}{\upsilon_0}\,\frac{\vartheta_4(\tau+1/4)+\vartheta_4(\tau-1/4)}{\vartheta_4(\tau+1/4)-\vartheta_4(\tau-1/4)} \tag{3.1}$$

and, therefore, $(\partial y/\partial\varphi)_{AD} = 0$, $y_{AD} = \mathrm{const}$.

Changing from theta-functions to elliptic functions in (3.1) [14], after some reduction and making the substitution

$$\xi=\frac{2\lambda_1\,\mathrm{sn}^2(2K\tau,\ k)-(1+\lambda_1)}{\lambda_1[1+\lambda_1-2\mathrm{sn}^2(2K\tau,\ k)]},\quad \lambda_1=\frac{1-k'}{1+k'}$$

we reduce expression (3.1) to a form which is the same as Eq. (7.8) in [1 p. 189] in which $\alpha = 0$.

2. *The case $d_1 = d_2 = 0$ (a non-submerged apron).* For this case we need merely put $B = C = 1$ in (2.6), and then $\Delta_B = 1$, $\Delta_C = \mathrm{dn}(2K\tau, k)$, $\lambda = k, f(\tau) = 2H$ and we obtain the results of [2].

3. *The case $\upsilon_0 = \rho$.* Taking the limit $\upsilon_0 \to \rho$ in the expression for $dz/d\tau$ in (2.6), using the formulae for α and β and expanding the interdeterminacy that arises, we find

$$\frac{dz}{d\tau} = \frac{f(\tau)}{\upsilon_0}\left(i - \frac{\pi}{K(k)Z[K(2\tau-1)]}\right) \tag{3.2}$$

where Z is the zeta-function [14].

4. *The case $\upsilon_0 > \rho$.* Transforming the expression for $dz/d\tau$ in (2.6) as it applies to the relation $\upsilon_0 > \rho$, we obtain

$$\frac{dz}{d\tau} = \frac{if(\tau)}{\upsilon_0}\frac{e^{-\pi\beta}\vartheta_3(\tau+\alpha i) - e^{\pi\beta}\vartheta_3(\tau-\alpha i)}{\vartheta_3(\tau+\alpha i) - \vartheta_3(\tau-\alpha i)} \tag{3.3}$$

$$\alpha = \pi^{-1}\,\mathrm{arth}\sqrt{(\upsilon_0-\rho)/(\upsilon_0+\rho)},\quad \beta = \pi^{-1}\,\mathrm{arth}\left(\sqrt{\upsilon_0^2-\rho^2}\,/\upsilon_0\right)$$

5. *The case $\upsilon_0 = \infty$ (flow past a tongue).* If the points B_1 and C_1 in the w plane merge at infinity (Fig. 4), the rectangle of the τ plane degenerates into the semi-strip $0 < \mathrm{Re}\,\tau < 1/2$, $0 < \mathrm{Im}\,\tau < \infty$ (Fig. 3b). Taking the limit in (3.3) as $k \to 0$, we find

$$\frac{dz}{d\tau} = \frac{2H}{\rho\sqrt{\Delta_B\Delta_C}}\frac{A+e^{2\pi\tau i}}{\sin 2\pi\tau} \tag{3.4}$$

where this time $\Delta_B = (1 - B^2)\sin^2\pi\tau + B^2$, $\Delta_C = 1 - (1 - C^2)\sin^2\pi, \tau$, $B = \sin \pi b$, $C = \sin \pi c$ and A is a certain constant which regularizes the position of the vertex of the cut G in the w plane.

Equation (3.4) is the same (apart from the notation) as Eq. (2.3) of [4], if we put $t = -\cos 2\pi\tau$, $C = A/\rho$, $n = 1/A$ in the latter. Moreover, if the points B and C merge, corresponding to the case of a point tongue, from (3.4) we obtain a relation which is the same as Eq. (2.5) of [5],with $\gamma = 0$ in the latter.

6. *The critical regime.* If the right-hand end of the boundary of separation butts up against the boundary of the tail, some of the salt water emerges on its surface. In the w plane this case corresponds to the cut disappearing and the circular pentagon degenerating into a triangle (for $\upsilon_0 \leq \rho$) or quadrilateral (when $\upsilon_0 > \rho$), bounded by two Apollo circles and straight-line sections orthogonal to those. The solution of the problem is obtained by taking the limit as $k' \to 0$ from (2.6) (in the first case) and from (3.3) for $\alpha = \Lambda/2$ (in the second).

4. CALCULATION OF THE PATTERN FLOW AND ANALYSIS OF THE NUMERICAL RESULTS

The basic representations (2.6), (3.1)–(3.4) contain three unknown constants: model k (or constant A in (3.4)) and two parameters of the mapping B and C, which can be found from the apron width l and the length of the vertical segments d_1 and d_2. These equations, integrated for different parts of the boundary of region τ, yield the parametric equations of the corresponding parts of the scheme. We can check by numerical means that the functions which appear in these equations are monotone, and thereby establish the unique solvability of the system relative to the unknown constants.

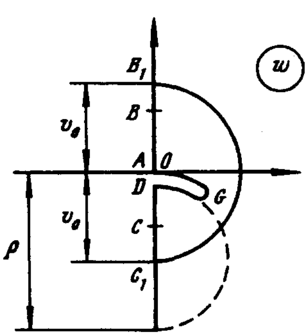

Fig. 4.

Having found the required parameters, we must determine the flow rate Q, the apron depth d, the difference T of the markers in the head and tail, and also calculate the coordinates of points of the underground contour of the apron B_1C_1 and the boundary of separation AD.

Figure 1 shows the underground contour of the apron and the boundary of separation calculated for $h_0 = 1.0$; $H = 0.16$; $l = 1.3$: $\rho = 0.175$; $v_0 = 0.0805$; $d_1 = 0.1$ and $d_2 = 0.11$. The results on the influence of the quantities H, ρ, v_0, l, d_1 and d_2 on the flow characteristics d, Q and T are summarized in Tables 1–3 (negative values of T denote that the boundary of the tail is above the abscissa axis). Each side of the tables corresponds to a different value of one of the parameters while the others are the same as in Fig. 1. The relation between the required characteristics and these parameters can be described as follows.

1. There is an interesting common relation between the flow characteristics and the parameters H and ρ: as the density ρ_2 of the salt water and the acting pressure H increase, the boundary of the tail drops, and the apron depth and, therefore, the flow rate Q increase. Thus, when H increases from 0.15 to 0.1675, the values of d and Q change by a factor of 1.5 and 2.7 respectively. Furthermore, as the parameter ρ decreases from 0.3 to 0.1, the depth h_2 is reduced by a factor of 3.7, so that the ratio h_1/h_2 increases from 1.7 to 9.0. Thus, the amount of squeezing increases as the pressure increases or as the density of the salt water is reduced.

2. On the other hand, as the parameters v_0, l and d_2 decrease, the apron is more deeply submerged and the flow rate increases. For example, as the width l decreases from 1.4 to 1.23, the depth d increases by a factor of 1.3, and the flow rate by a factor of 1.7. This confirms one important result of [3]: the shorter the apron, the thicker it must be for the same velocity v_0.

3. The quantity T changes the most: when the velocity v_0 increases by no more than 6.2%, T changes by a factor of 6.4.

4. The influence of the parameters d_1 and d_2 on the flow pattern is especially interesting. When 174i and d_1 and d_2 increase from 0.05 to 0.4, the flow rate, which, as we can see from Table 3, is practically the same for equal values of these parameters, decreases by 38–39%, the apron depth d increasing by 18% in the first case and decreasing by 24% in the second. The parameter which has the greatest influence on T is d_2. Thus, for $d_2 = 0.05$; 0.2 and d_2 and 0.4 we have, respectively, $T = -0.0034$; 0.0219 and -0.0253, showing that the change is non-monotone. A graph of T versus d_2/d_1 is shown in Fig. 5.

Strangely enough, as the calculations show, this problem can yield a very interesting pattern flow, due to the fact that close to the vertical wall C_1C when $x < l_2, y_{B1C1}(x) > 0$. This can be interpreted as an apron which has a "streamlined" tongue or tooth in its lower part, when a similar scheme to that shown in Fig. 139b of [1] can be used.

In such cases the difference between the markers in the head and tail might be quite considerable. Thus, when $l = 1.4$, we have $T = -0.1334$ (see the right-hand side of Table 2), so that $|T/d_2| = 1.21$. This ratio increases with the apron width l.

Table 1

$H \times 10^2$	$d \times 10^3$	$Q \times 10^4$	$T \times 10^4$	ρ	$d \times 10^3$	$Q \times 10^4$	$T \times 10^4$
15.00	427	480	−1038	0.1	518	708	−800
16.00	564	757	176	0.3	585	768	564
16.75	664	1283	914	∞	609	773	1035

Table 2

$v_0 \times 10^4$	$d \times 10^3$	$Q \times 10^4$	$T \times 10^4$	l	$d \times 10^3$	$Q \times 10^4$	$T \times 10^4$
805	564	757	176	1.23	645	1559	1024
835	483	600	−556	1.32	538	688	−95
855	428	518	−1125	1.40	440	495	−1334

Table 3

d_1	$d \times 10^3$	$Q \times 10^4$	$T \times 10^4$	d_2	$d \times 10^3$	$Q \times 10^4$	$T \times 10^4$
0.05	565	812	173	0.05	585	823	−34
0.20	577	649	329	0.20	524	659	219
0.40	669	502	1188	0.40	441	499	−253

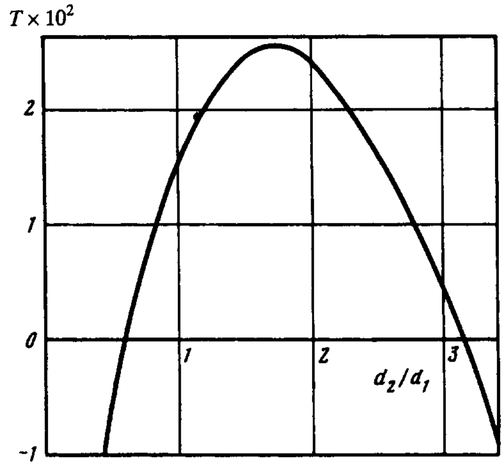

Fig. 5.

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